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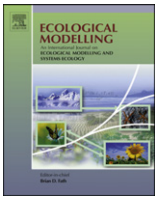

Letter to the Editor

Comments to "The effect of prey refuge in a simple predator–prey model" [Ecol. Model. 222 (September(18)) (2011) 3453–3454]

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ABSTRACT

In this work we comment on the letter to the editor "The effect of prey refuge in a simple predator–prey model" with observations to the article by González-Olivares and Ramos-Jiliberto published in 2003.

1. Introduction

In the paper studied in González-Olivares and Ramos-Jiliberto (2003), a modification of the Rosenzweig-MacArthur model (Turchin, 2003) is considered, supposing that there exists a quantity x_r of the prey population that occupies a refuge.

Considering that the quantity of refuged prey depends on the capacity of hiding places found in the environment, then $x_r = \gamma$, i.e., the quantity of prey in refuge is a constant quantity γ of the population (González-Olivares and Ramos-Jiliberto, 2003).

The work of González-Olivares and Ramos-Jiliberto (2003) was aimed mainly at showing that a fraction of the prey population in refuge would imply oscillatory behavior of the model dynamics, responding to the claim that refuges have a stabilizing effect on that dynamics.

That paper left some open problems which have subsequently been resolved and some results have been presented with more explicit relationships. For example, in Chen et al. (2010) the global stability and unstability properties of the equilibria and the existence and uniqueness of limit cycles of model (1) are considered.

It has also been used to analyze models that are considered constant harvesting of the prey population (Ji and Wu, 2010) or a proportion of both population (Tao et al., 2011).

Now, in Ma et al. (2011) the authors affirm that: "the theorem 4.3a (in González-Olivares and Ramos-Jiliberto, 2003) forfeit the ecological meanings in their work. In this sight, we think that it is necessary to correct the their results in order to have ecological meanings and find the stabilizing effect clearly."

To reply the comments expressed in Ma et al. (2011) we reiterate some calculations.

2. Model with a constant number of prey in refuges

The model is described by the system

$$X_\eta : \begin{cases} \dfrac{dx}{dt} = r\left(1 - \dfrac{x}{K}\right)x - \dfrac{q(x-\gamma)}{x-\gamma+a}y \\ \dfrac{dy}{dt} = \left(\dfrac{p(x-\gamma)}{x-\gamma+a} - c\right)y \end{cases} \tag{1}$$

where $x = x(t)$ and $y = y(t)$ indicate the prey and predator population sizes respectively for $t > 0$, measure as number of individuals, density or biomass. The parameters are all positives, i.e., $\eta = (r, K, q, a, p, c, \gamma) \in \mathbb{R}^7_+$, having the following meanings (see González-Olivares and Ramos-Jiliberto, 2003; Ma et al., 2011):

r is the intrinsic per capita growth rate of prey,
K is the prey environmental carrying capacity,
q is the maximal per capita consumption rate of predators,
a is the amount of prey needed to achieve one-half of q,
c is the per capita death rate of predators,
p is the efficiency with which predators convert consumed prey into new predators, and
γ is a constant quantity of prey using refuges.

For ecological reasons $a < K$.

The intersection of isoclines $\frac{p(x-\gamma)}{x-\gamma+a} - c = 0$ and $r\left(1 - \frac{x}{K}\right)x - \frac{q(x-\gamma)}{x-\gamma+a}y = 0$, determine the equilibrium point $(x_e, y_e) = \left(\frac{ac+(p-c)\gamma}{(p-c)}, \frac{rp}{qc}\left(1 - \frac{x_e}{K}\right)x_e\right)$.

Clearly, (x_e, y_e) lies inside of the first quadrant (with positive population size for both prey and predators), if and only if, $p - c > 0$ and $\frac{ac+(p-c)\gamma}{(p-c)} < K$, i.e., if and only if, $p > c$ and $\gamma < \frac{(p-c)K-ac}{(p-c)}$.

In order to simplify the calculus, we made a reparameterization of the vector field X_η or system (1), doing a change of variables and a time rescaling, given by the function

$$\varphi(u, v, \tau) = \left(Ku, \frac{rK}{q}v, \frac{r}{u-L+A}\tau\right),$$

which is a diffeomorphism (Chicone, 2006). For this reason the vector field X_μ, is topologically equivalent to the vector field $Z_\mu = \varphi \circ X_\mu$ with $Z_\mu = P(u,v)\frac{\partial}{\partial u} + Q(u,v)\frac{\partial}{\partial v}$ and the associated differential equation system is given by the polynomial system of third degree.

$$Z_\mu : \begin{cases} \dfrac{du}{d\tau} = (1-u)u(u-L+A) - (u-L)v \\ \dfrac{dv}{d\tau} = (u - L - C(u-L+A))v \end{cases}, \tag{2}$$

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where $\mu = (A, B, C, L) \in \mathbb{R}_+^4$, with $A = \frac{a}{K}$, $B = \frac{p}{r}$, $C = \frac{c}{p}$ and $L = \frac{\gamma}{K}$. Thus, the behavior of system (2) is qualitatively equivalent to system (1).

The equilibrium points are (0, 0), (1, 0), $(L - A, 0)$ and (u_e, v_e) which lies in the isoclines

$$u - L - C(u + A - L) = 0 \quad \text{and} \quad r(1 - u)u(u - L + A) - (u - L)v = 0$$

obtaining the positive community equilibrium

$$u_e = \frac{L + C(A - L)}{1 - C} \quad \text{and} \quad v_e = \frac{r(1 - u_e)u_e}{C}.$$

3. The revision

The revision will be made mainly for the unique positive equilibrium point. The Jacobian matrix of system (2) is

$$DZ_\mu(u, v) = \begin{pmatrix} DZ_\mu(u, v)_{11} & -(u - L) \\ Bv(1 - C) & B(u - L - C(u + A - L)) \end{pmatrix},$$

with $DZ_\mu(u, v)_{11} = -3u^2 + 2(L - A + 1)u + (A - L - v)$.

Evaluated in (u_e, v_e) it is obtained

$$DZ_\mu(u_e, v_e) = \begin{pmatrix} -\dfrac{S}{C(1 - C)^2} & -\dfrac{AC}{1 - C} \\ Bv_e(1 - C) & 0 \end{pmatrix}$$

being

$$S = (1 - C)^3 L^2 - (1 - C)((1 - C)^2 + 2AC^2)L + AC^2(1 - A - C - AC). \tag{3}$$

Conditions for the existence and nonexistence of at least one limit cycle around the positive equilibrium point (u_e, v_e) are written in terms of the ancillary parameter S (Theorem 4.3a) in González-Olivares and Ramos-Jiliberto (2003). Then,

(a) If $S > 0$, the singularity (u_e, v_e) of system (2) is an unstable equilibrium point, surrounded by a limit cycle.
(b) If $S < 0$, the singularity (u_e, v_e) of system (2) is a locally asymptotically stable equilibrium point.

Rewriting expression (3) in terms of the original parameters and after simple calculations, we obtain S' given by

$$S' = (p - c)^3\gamma^2 - (p - c)(2ac^2 + K(p - c)^2)\gamma + ac^2(K(p - c) - a(c + p)). \tag{4}$$

We note that, if $S' = 0$, the quadratic equation (4) for γ has two solutions given by:

$$\gamma_{1'} = \frac{2ac^2 + K(p - c)^2 - \sqrt{\Delta}}{2(p - c)^2} \qquad \gamma_{2'} = \frac{2ac^2 + K(p - c)^2 + \sqrt{\Delta}}{2(p - c)^2}, \tag{5}$$

where $\Delta = (p - c)^4 K^2 + 4a^2c^2p^2$.

Unfortunately, in Theorem 4.3a (González-Olivares and Ramos-Jiliberto, 2003), there exists a mistake in the last sign of expression (4), appearing as minus (−), which has no influence on the statement of the theorem.

Considering the erroneous expression for S', it is obtained that

$$S' = (p - c)((p - c)^2\gamma^2 - (K(p - c)^2 + 2ac^2)\gamma + ac^2(K + a));$$

so, the sign of S' depends on the sign of the factor

$$S'' = (p - c)^2\gamma^2 - (K(p - c)^2 + 2ac^2)\gamma + ac^2(K + a),$$

which is also, a quadratic equation for γ. If $S'' = 0$, it is obtained that,

$$\gamma_{1''} = \frac{K(p - c)^2 + 2ac^2 - \sqrt{\Delta_1}}{2(p - c)^2} \qquad \gamma_{2''} = \frac{K(p - c)^2 + 2ac^2 + \sqrt{\Delta_1}}{2(p - c)^2},$$

where $\Delta_1 = (K(p - c)^2 + 2ac^2)^2 - 4(p - c)^2ac^2(K + a)$.

3.1. Our reply

The authors of the note (Ma et al., 2011) assure the existence of two values γ_1 and γ_2 defined by

$$\gamma_1 = \frac{K}{2} + \frac{p}{ac} - \frac{ac}{p - c} - \sqrt{\nabla} \qquad \gamma_2 = \frac{K}{2} + \frac{p}{ac} - \frac{ac}{p - c} + \sqrt{\nabla}, \tag{6}$$

with $\nabla = \left(\frac{K}{2}\right)^2 + \left(\frac{p}{ac}\right)^2$ and $\gamma_1 < \gamma_2$. They say that by simple computation, they found that:

(i) inequation $S' > 0$ is equivalent to the inequations $\gamma < \gamma_1$ or $\gamma > \gamma_2$.
(ii) inequation $S' < 0$ is equivalent to the inequations $\gamma_1 < \gamma < \gamma_2$.

As we could not obtain the relations (6), we will try to confirm this stated equivalence, assuming that they are solutions of a quadratic polynomial $p(u)$; then, it has

$$p(u) = \left(\gamma - \left(\frac{K}{2} + \frac{p}{ac} - \frac{ac}{p - c} + \sqrt{\nabla}\right)\right) \times \left(\gamma - \left(\frac{K}{2} + \frac{p}{ac} - \frac{ac}{p - c} - \sqrt{\nabla}\right)\right).$$

Therefore,

$$p(u) = \gamma^2 + \frac{-ac(p - c)K + 2(a^2c^2 - p(p - c))}{ac(p - c)}\gamma + \frac{(p - c)(p(p - c) - a^2c^2)K - ac(2p(p - c) - a^2c^2)}{ac(p - c)^2}$$

or

$$p(u) = \frac{ac(p - c)^2\gamma^2 + (p - c)(-ac(p - c)K + 2(a^2c^2 - p(p - c)))\gamma + (p - c)(p(p - c) - a^2c^2)K - ac(2p(p - c) - a^2c^2)}{ac(p - c)^2}$$

Clearly, the equation $p(u) = 0$ is not equivalent neither to the correct equation $S' = 0$ nor to the incorrect relation $S'' = 0$, if the authors of (Ma et al., 2011) had considered the sign error.

Although we found interesting that the theorems were expressed in terms of the parameter γ, will not comment the statements of Propositions 1 and 2 in Ma et al. (2011), since the solutions of $p(u) = 0$ are bounded by arbitrary values, which we could not obtain by a simple calculation in order to prove the mentioned equivalence of $p(u)$ with S'.

However, these propositions can be improved by considering the values of γ given by us in relation (5).

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